For a good introduction to nonlinear dynamics, see (Strogatz and Dichter 2016). A more engineer-oriented introduction is (Kolk and Lerman 1993).

9.1 Nonlinear State-Space Models

A state-space model has the general form

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) \tag{9.1}$$

$$\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t) \tag{9.2}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are vector-valued functions that depend on the system. **Nonlinear state-space models** are those for which $\boldsymbol{f}$ is a nonlinear functional of either $\boldsymbol{x}$ or $\boldsymbol{u}$. For instance, a state variable x_1 might appear as x_1^2 or two state variables might combine as $x_1 x_2$ or an input u_1 might enter the equations as $\log u_1$.

9.1.1 Autonomous and Nonautonomous Systems

An **autonomous system** is one for which $\boldsymbol{f}(\boldsymbol{x})$, with neither time nor input appearing explicitly. A **nonautonomous system** is one for which either t or $\boldsymbol{u}$ *do* appear explicitly in $\boldsymbol{f}$. It turns out that we can always write nonautonomous systems as autonomous by substituting in $\boldsymbol{u}(t)$ and introducing an extra state variable for t (Strogatz and Dichter 2016).

Therefore, without loss of generality, we will focus on ways of analyzing autonomous systems.

9.1.2 Equilibrium

An **equilibrium state** (also called a stationary point) $\overline{\boldsymbol{x}}$ is one for which $d\boldsymbol{x}/dt = \boldsymbol{0}$. In most cases, this occurs only when the input $\boldsymbol{u}$ is a constant $\overline{\boldsymbol{u}}$ and, for time-varying systems, at a given time $\overline{t}$. For autonomous systems, equilibrium occurs when the following holds:

$$\boldsymbol{f}(\overline{\boldsymbol{x}}) = \boldsymbol{0}.$$

This is a system of nonlinear algebraic equations, which can be challenging to solve for $\overline{\boldsymbol{x}}$. However, frequently, several solutions—that is, equilibrium states—do exist.

9.2 Nonlinear System Characteristics

Characterizing nonlinear systems can be challenging without the tools developed for linear system characterization. However, there are ways of characterizing nonlinear systems, and we'll here explore a few.

9.2.1 Those In-Common with Linear Systems

As with linear systems, the **system order** is either the number of state-variables required to describe the system or, equivalently, the highest-order derivative in a single scalar differential equation describing the system.

Similarly, nonlinear systems can have state variables that depend on time alone or those that also depend on space (or some other independent variable). The former lead to ordinary differential equations (ODEs) and the latter to partial differential equations (PDEs).

Equilibrium was already considered in section 9.1.2.

9.2.2 Stability

In terms of system performance, perhaps no other criterion is as important as stability.

Definition 9.1

If $\boldsymbol{x}$ is perturbed from an equilibrium state $\overline{\boldsymbol{x}}$, the response $\boldsymbol{x}(t)$ can:

1. asymptotically return to $\overline{\boldsymbol{x}}$ (asymptotically stable),
2. diverge from $\overline{\boldsymbol{x}}$ (unstable), or
3. remain perturned or oscillate about $\overline{\boldsymbol{x}}$ with a constant amplitude (marginally stable).

Notice that this definition is actually local: stability in the neighborhood of one equilibrium may not be the same as in the neighborhood of another.

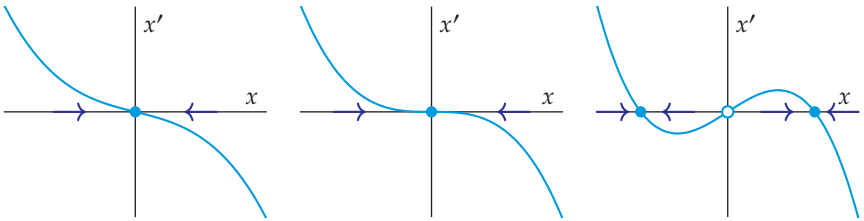
Other than nonlinear systems' lack of linear systems' eigenvalues, poles, and roots of the characteristic equation from which to compute it, the primary difference between the stability of linear and nonlinear systems is that nonlinear system stability is often difficult to establish globally. Using a linear system's eigenvalues, it is straightforward to establish stable, unstable, and marginally stable subspaces of state-space (via transforming to an eigenvector basis). For nonlinear systems, no such method exists. However, we are not without tools to explore nonlinear system stability. One mathematical tool to consider is Lyapunov stability theory, which is beyond the scope of this course, but has good treatments in (Brogan 1991; Ch. 10) and (Choukchou-Braham et al. 2013; App. A).

9.2.3 Qualities of Equilibria

Equilibria (i.e. stationary points) come in a variety of qualities. It is instructive to consider the first-order differential equation in state variable x with real constant r:

$$x' = rx - x^3.$$

If we plot x' versus x for different values of r, we obtain the plots of figure 9.1.



(a) $r < 0$ (b) $r = 0$ (c) $r > 0$

Figure 9.1. Plots of x' versus x for section 9.2.3.

By definition, equilibria occur when $x' = 0$, so the x-axis crossings of figure 9.1 are equilibria. The blue arrows on the x-axis show the direction (sign) of state change x', quantified by the plots. For both (a) and (b), only one equilibrium exists: $x = 0$. Note that the blue arrows in both plots point *toward* the equilibrium. In such cases—that is, when a neighborhood exists around an equilibrium for which state changes point toward the equilibrium—the equilibrium is called an **attractor** or **sink**. Note that attractors are stable.

Now consider (c) of figure 9.1. When $r > 0$, three equilibria emerge. This change of the number of equilibria with the changing of a parameter is called a **bifurcation**. A plot of bifurcations versus the parameter is called a **bifurcation diagram**. The $x = 0$ equilibrium now has arrows that point away from it. Such an equilibrium is called a **repeller** or **source** and is unstable. The other two equilibria here are (stable) attractors. Consider a very small initial condition $x(0) = \epsilon$. If $\epsilon > 0$, the repeller pushes away x and the positive attractor pulls x to itself. Conversely, if $\epsilon < 0$, the repeller again pushes away x and the negative attractor pulls x to itself.

Another type of equilibrium is called the saddle: one which acts as an attractor along some lines and as a repeller along others. We will see this type in the following example.

Example 9.1

Consider the dynamical equation

$$x' = x^2 + r$$

with r a real constant. Sketch x' vs x for negative, zero, and positive r. Identify and classify each of the equilibria.

TODO

9.3 Simulating Nonlinear Systems

Example 9.2

Simulate a nonlinear unicycle in Python.

First, load some Python packages.

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt
```

The state equation can be encoded via the following function `f`.

```
def f(t, x, u, c):
  dxdt = [
    x[3]*np.cos(x[2]),
    x[3]*np.sin(x[2]),
    x[4],
    1/c[0] * u(t)[0],
    1/c[1] * u(t)[1]
  ]
  return dxdt
```

The input function `u` must also be defined.

```
def u(t):
  return [
    15*(1+np.cos(t)),
    25*np.sin(3*t)
  ]
```

Define time spans, initial values, and constants

```
tspan = np.linspace(0, 50, 300)
xinit = [0,0,0,0,0]
mass = 10
inertia = 10
c = [mass,inertia]
```