7.2 Boundary Value Problems

2H

Before we introduce an important solution method for PDEs in section 7.3, we consider an *ordinary* differential equation that will arise in that method when dealing with a single spatial dimension x: the **sturm-liouville (S-L) differential equation**. Let p, q, σ be functions of x on open interval (a, b). Let X be the dependent variable and λ constant. The **regular S-L problem** is the S-L ODE[2]

$$\frac{d}{dx}(pX') + qX + \lambda\sigma X = 0 \tag{7.2}$$

with boundary conditions

$$\beta_1 X(a) + \beta_2 X'(a) = 0 \tag{7.3}$$

$$\beta_3 X(b) + \beta_4 X'(b) = 0 \tag{7.4}$$

with coefficients $\beta_i \in \mathbb{R}$. This is a type of **boundary value problem**.

This problem has nontrivial solutions, called **eigenfunctions** $X_n(x)$ with $n \in \mathbb{Z}_+$, corresponding to specific values of $\lambda = \lambda_n$ called **eigenvalues**.[3] There are several important theorems proven about this (see (Haberman 2018; § 5.3)). Of greatest interest to us are that

1. There exist an infinite number of eigenfunctions X_n (unique within a multiplicative constant)
2. There exists a unique corresponding *real* eigenvalue λ_n for each eigenfunction X_n
3. The eigenvalues can be ordered as $\lambda_1 < \lambda_2 < \cdots$
4. Eigenfunction X_n has $n - 1$ zeros on open interval (a, b)
5. The eigenfunctions X_n form an orthogonal basis with respect to weighting function σ such that any piecewise continuous function $f : [a, b] \to \mathbb{R}$ can be represented by a generalized fourier series on $[a, b]$

This last theorem will be of particular interest in section 7.3.

2. For the S-L problem to be *regular*, it has the additional constraints that p, q, σ are continuous and $p, \sigma > 0$ on $[a, b]$. This is also sometimes called the sturm-liouville eigenvalue problem. See (Haberman 2018; § 5.3) for the more general (non-regular) S-L problem and (§ 7.4) for the multi-dimensional analog.

3. These eigenvalues are closely related to, but distinct from, the "eigenvalues" that arise in systems of linear ODEs.

7.2.1 Types of Boundary Conditions

Boundary conditions of the sturm-liouville kind equation (7.3) have four sub-types:

Dirichlet for just $\beta_2, \beta_4 = 0$,
Neumann for just $\beta_1, \beta_3 = 0$,
Robin for all $\beta_i \neq 0$, and
Mixed if $\beta_1 = 0$, $\beta_3 \neq 0$; if $\beta_2 = 0$, $\beta_4 \neq 0$.

There are many problems that are *not* regular sturm-liouville problems. For instance, the right-hand sides of equation (7.3) are zero, making them **homogeneous boundary conditions**; however, these can also be nonzero. Another case is **periodic boundary conditions**:

$$X(a) = X(b) \tag{7.5}$$

$$X'(a) = X'(b). \tag{7.6}$$

Example 7.1

Consider the differential equation

$$X'' + \lambda X = 0$$

with dirichlet boundary conditions on the boundary of the interval $[0, L]$

$$X(0) = 0 \quad \text{and} \quad X(L) = 0.$$

Solve for the eigenvalues and eigenfunctions.

This is a sturm-liouville problem, so we know the eigenvalues are real. The well-known general solution to the ODE is

$$X(x) = \begin{cases} k_1 + k_2 x & \lambda = 0 \\ k_1 e^{j\sqrt{\lambda}x} + k_2 e^{-j\sqrt{\lambda}x} & \text{otherwise} \end{cases}$$

with real constants k_1, k_2. The solution must also satisfy the boundary conditions. Let's apply them to the case of $\lambda = 0$ first:

$$X(0) = 0 \Rightarrow k_1 + k_2(0) = 0 \Rightarrow k_1 = 0$$

$$X(L) = 0 \Rightarrow k_1 + k_2(L) = 0 \Rightarrow k_2 = -k_1/L.$$

Together, these imply $k_1 = k_2 = 0$, which gives the *trivial solution* $X(x) = 0$, in which we aren't interested. We say, then, that for nontrivial solutions, $\lambda \neq 0$. Now let's check $\lambda < 0$. The solution becomes

$$\begin{aligned} X(x) &= k_1 e^{-\sqrt{|\lambda|}x} + k_2 e^{\sqrt{|\lambda|}x} \\ &= k_3 \cosh(\sqrt{|\lambda|}x) + k_4 \sinh(\sqrt{|\lambda|}x) \end{aligned}$$

where k_3 and k_4 are real constants. Again applying the boundary conditions:

$$X(0) = 0 \Rightarrow k_3 \cosh(0) + k_4 \sinh(0) = 0 \Rightarrow k_3 + 0 = 0 \Rightarrow k_3 = 0$$

$$X(L) = 0 \Rightarrow 0 \cosh(\sqrt{|\lambda|}L) + k_4 \sinh(\sqrt{|\lambda|}L) = 0 \Rightarrow k_4 \sinh(\sqrt{|\lambda|}L) = 0.$$

However, $\sinh(\sqrt{|\lambda|}L) \neq 0$ for $L > 0$, so $k_4 = k_3 = 0$—again, the trivial solution. Now let's try $\lambda > 0$. The solution can be written

$$X(x) = k_5 \cos(\sqrt{\lambda}x) + k_6 \sin(\sqrt{\lambda}x).$$

Applying the boundary conditions for this case:

$$X(0) = 0 \Rightarrow k_5 \cos(0) + k_6 \sin(0) = 0 \Rightarrow k_5 + 0 = 0 \Rightarrow k_5 = 0$$

$$X(L) = 0 \Rightarrow 0 \cos(\sqrt{\lambda}L) + k_6 \sin(\sqrt{\lambda}L) = 0 \Rightarrow k_6 \sin(\sqrt{\lambda}L) = 0.$$

Now, $\sin(\sqrt{\lambda}L) = 0$ for

$$\sqrt{\lambda}L = n\pi \Rightarrow$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2. \qquad (n \in \mathbb{Z}_+)$$

Therefore, the only nontrivial solutions that satisfy both the ODE and the boundary conditions are the *eigenfunctions*

$$X_n(x) = \sin\left(\sqrt{\lambda_n}x\right) \tag{7.7}$$

$$= \sin\left(\frac{n\pi}{L}x\right) \tag{7.8}$$

with corresponding *eigenvalues*

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

Note that because $\lambda > 0$, λ_1 is the lowest eigenvalue.

Plotting the Eigenfunctions

```
import numpy as np
import matplotlib.pyplot as plt
```

Set $L = 1$ and compute values for the first four eigenvalues `lambda_n` and eigenfunctions `X_n`.

```
L = 1
x = np.linspace(0, L, 100)
n = np.linspace(1, 4, 4, dtype=int)
lambda_n = (n*np.pi/L)**2
X_n = np.zeros([len(n), len(x)])
for i,n_i in enumerate(n):
  X_n[i, :] = np.sin(np.sqrt(lambda_n[i])*x)
```

Plot the eigenfunctions.

```
fig, ax = plt.subplots()
for i, n_i in enumerate(n):
  ax.plot(x, X_n[i,:], linewidth=2,label='$n = '+str(n_i)+'$')
plt.legend()
plt.show()
```

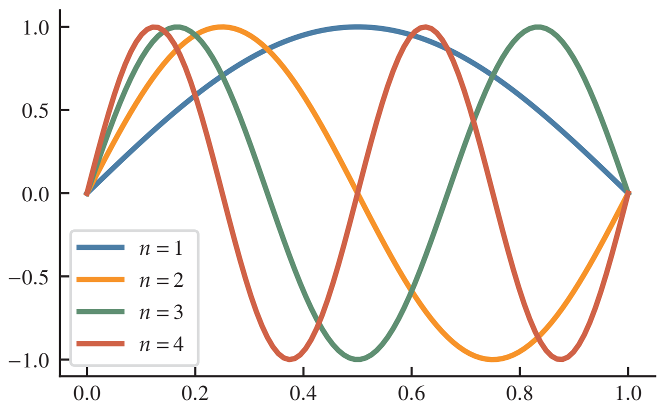

Figure 7.1. Eigenfunctions $X_n(x)$.

We see that the fourth of the S-L theorems appears true: $n - 1$ zeros of X_n exist on the open interval $(0, 1)$.

7.2.2 Irregular Sturm-Liouville Problems

Sturm-Liouville problems that do not satisfy the regularity conditions are called **irregular**. The nice theorems for regular S-L problems may not hold for irregular S-L problems. However, we can often still solve irregular S-L problems using the same methods as for regular problems.

Example 7.2

Consider the ODE called **Bessel's equation** (Kreyszig 2011; § 5.4), for real independent variable s and dependent variable $y(s)$,

$$s^2 y'' + s y' + \left(s^2 - \nu^2\right) y = 0.$$

This ODE arises in polar coordinate PDE models of circular membranes, where y is the membrane displacement and $s = kr$, and where r is the radius of the membrane and k is a constant sometimes called the wavenumber.

Consider the following boundary conditions:

- At radius $r = R$, the membrane is fixed, so $y = 0$.
- At radius $r = 0$, the membrane is free to move, but the displacement is finite, so $|y| < \infty$.

Prove that the Bessel equation is an irregular Sturm-Liouville problem and solve for the eigenvalues and eigenfunctions for the case $\nu = 0$.

We proceed to show that the Bessel equation with the given boundary conditions is an irregular Sturm-Liouville problem by first showing that the Bessel equation can be written in the form of a Sturm-Liouville problem ODE, then showing that the boundary conditions are not regular. Dividing the Bessel equation by s gives

$$s y'' + y' + \frac{s^2 - \nu^2}{s} y = 0 \implies \qquad (z \neq 0)$$

$$\frac{d}{ds}\left(s y'\right) + \left(s - \nu^2/s\right) y = 0. \tag{7.9}$$

So we see that this is equivalent to the Sturm-Liouville problem's ODE,

$$\frac{d}{dx}\left(pX'\right) + (\lambda\sigma + q)\, X = 0, \tag{7.10}$$

with $x = s$, $X = y$, $p(s) = s$, $q(s) = -\nu^2/s$, $\sigma(s) = s$, and $\lambda = 1$.

Now let us consider the boundary conditions. The second boundary condition cannot be written as a linear combination of y and y' at $s = 0$, therefore this is an irregular Sturm-Liouville problem.

Solutions for Bessel's equation are Bessel functions of the first kind, $J_\nu(s)$ (section 6.4), and Bessel functions of the second kind, $Y_\nu(s)$ (§ 5.5). For $\nu = 0$, the Bessel equation simplifies to

$$s^2 y'' + s y' + s^2 y = 0$$

and the solutions are zeroth-order Bessel functions of the first kind, $J_0(s)$, and of the second kind, $Y_0(s)$. That is, the general solution is

$$y(s) = a J_0(s) + b Y_0(s).$$

However, $Y_0(s)$ is singular at $s = 0$, so the boundary condition $|y(0)| < \infty$ requires $b = 0$. Therefore, the eigenfunctions are

$$y_n(s) = J_0(s) = J_0(k_n r)$$

for eigenvalues $\lambda_n = k_n^2$ and $n \in \mathbb{Z}_+$. On the boundary, $y_n(kR) = 0$ implies that $k_n R$ are the zeros of $J_0(s)$, what we called $\alpha_{0,n}$ in example 6.4. Therefore, the eigenvalues are

$$\lambda_n = \left(\frac{\alpha_{0,n}}{R}\right)^2.$$

Plotting the Eigenfunctions We proceed in Python. First, load packages.

```
import numpy as np
import sympy as sp
import scipy
from scipy.special import jn_zeros
import matplotlib.pyplot as plt
```

The eigenvalues can be computed from the zeros of the Bessel function zeros $\alpha_{0,n}$, where $n = 1, 2, 3, \ldots$ as follows:

```
n = sp.symbols('n', integer=True, positive=True)
k, R = sp.symbols('k, R', positive=True, real=True)
r = sp.symbols('r', nonnegative=True)
J_0 = sp.besselj(0, k * r)
N_zeros = 5
alpha_0_n = jn_zeros(0, N_zeros)
params = {R: 1}  # Set R = 1
lambda_n_ = (alpha_0_n / params[R])**2
print(f"The first {N_zeros} eigenvalues are:")
print(lambda_n_)
```

```
The first 5 eigenvalues are:
[  5.78318596  30.47126234  74.88700679 139.04028443 222.93230362]
```

The eigenfunctions are given by the Bessel functions of the first kind $J_0(k_n r)$.

```
def k_n(lambda_n): return np.sqrt(lambda_n)
k_n_ = k_n(lambda_n_)
```

Plot the eigenfunctions.

```
r_plt = np.linspace(0, 1, 101)
print(J_0)
y_n_fun = sp.lambdify((r, k), J_0, modules=['numpy', 'scipy'])
fig, ax = plt.subplots()
for i in range(5):
    k_i = k_n_[i]
    ax.plot(r_plt, y_n_fun(r_plt, k_i), label=f'$y_{i+1}(k_{i+1} r)$')
ax.spines['bottom'].set_position('zero')
ax.spines['left'].set_position('zero')
ax.set_xlabel('$r$')
ax.set_ylabel('Eigenfunctions $y_n(k_n r)$')
ax.legend()
plt.show()
```

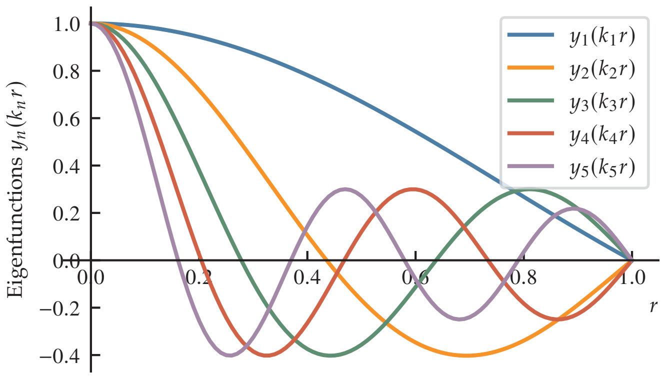

Figure 7.2. Eigenfunctions for the first few eigenvalues.

The plot shows the eigenfunctions $y_n(k_n r)$ for the first few eigenvalues. Note that the boundary conditions are satisfied and that the eigenfunctions appear to be radial modes of vibration for a circular membrane.

7.3 PDE Solution by Separation of Variables

AN

We are now ready to learn one of the most important techniques for solving PDEs: **separation of variables**. It applies only to **linear** PDEs since it will require the principle of superposition. Not all linear PDEs yield to this solution technique, but several that are important do.

The technique includes the following steps.

assume a product solution Assume the solution can be written as a **product solution** u_p: the product of functions of each independent variable.

separate PDE Substitute u_p into the PDE and rearrange such that at least one side of the equation has functions of a single independent variabe. If this is possible, the PDE is called **separable**.

set equal to a constant Each side of the equation depends on different independent variables; therefore, they must each equal the same constant, often called $-\lambda$.

repeat separation, as needed If there are more than two independent variables, there will be an ODE in the separated variable and a PDE (with one fewer variables) in the other independent variables. Attempt to separate the PDE until only ODEs remain.

solve each boundary value problem Solve each boundary value problem ODE, ignoring the initial conditions for now.

solve the time variable ODE Solve for the general solution of the time variable ODE, sans initial conditions.

construct the product solution Multiply the solution in each variable to construct the product solution u_p. If the boundary value problems were sturm-liouville, the product solution is a family of **eigenfunctions** from which any function can be constructed via a generalized fourier series.

apply the initial condition The product solutions individually usually do not meet the initial condition. However, a generalized fourier series of them nearly always does. **Superposition** tells us a linear combination of solutions to the PDE and boundary conditions is also a solution; the unique series that also satisfies the initial condition is the unique solution to the entire problem.

Example 7.3

Consider the one-dimensional diffusion equation PDE[a]

$$\partial_t u(t,x) = k\partial^2_{xx} u(t,x)$$

with real constant k, with dirichlet boundary conditions on inverval $x \in [0, L]$

$$u(t,0) = 0 \tag{7.11}$$

$$u(t,L) = 0, \tag{7.12}$$